Introduction to Data Mining

Link Analysis-1

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In This Lecture

- Motivation for link analysis
- Pagerank: an important graph ranking algorithm
  - Flow and random walk formulation

Outline

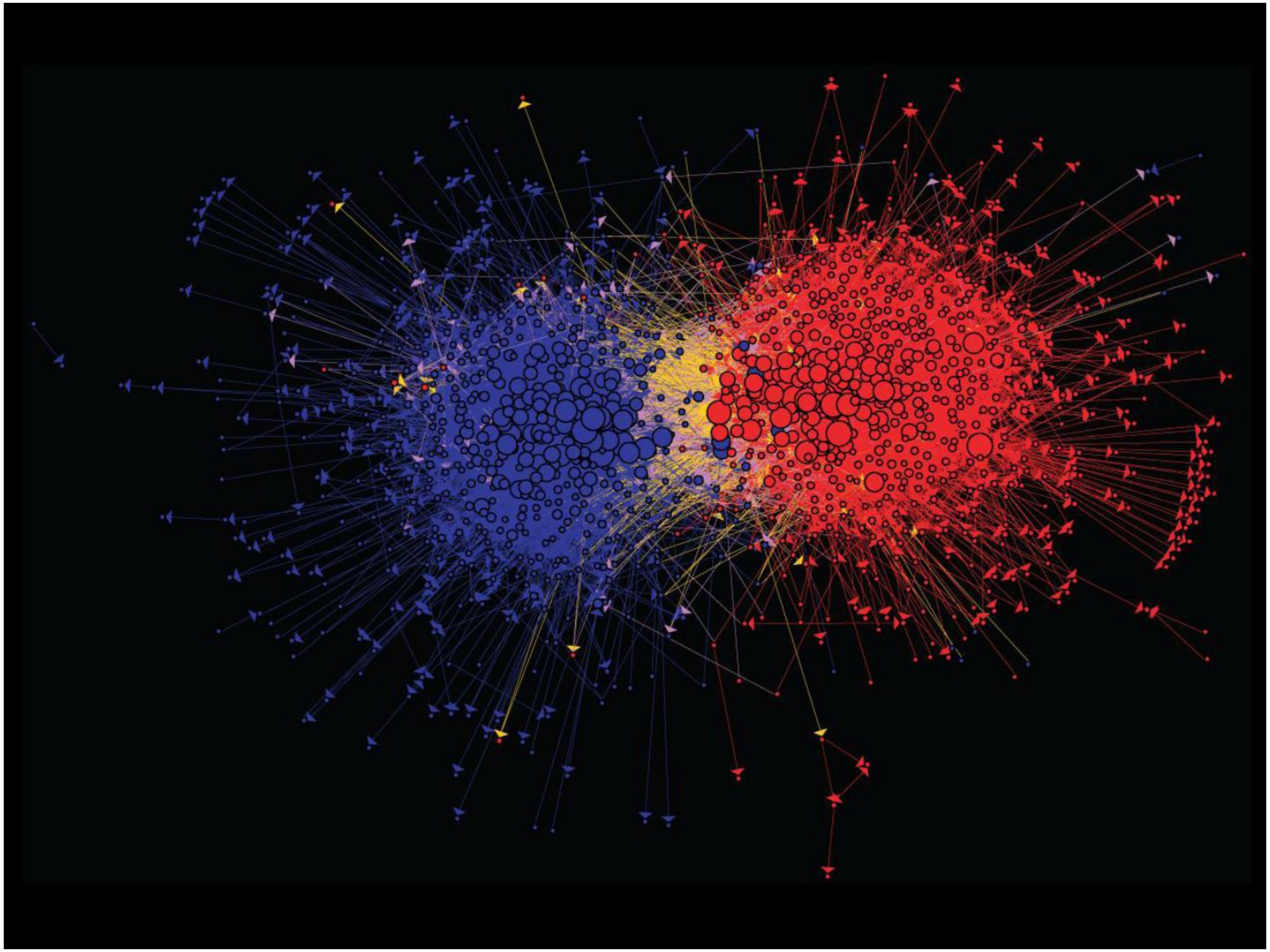
- ☐ **Overview**
- ☐ PageRank: Flow Formulation

Graph Data: Social Networks

Facebook social graph

4-degrees of separation [Backstrom-Boldi-Rosa-Ugander-Vigna, 2011]

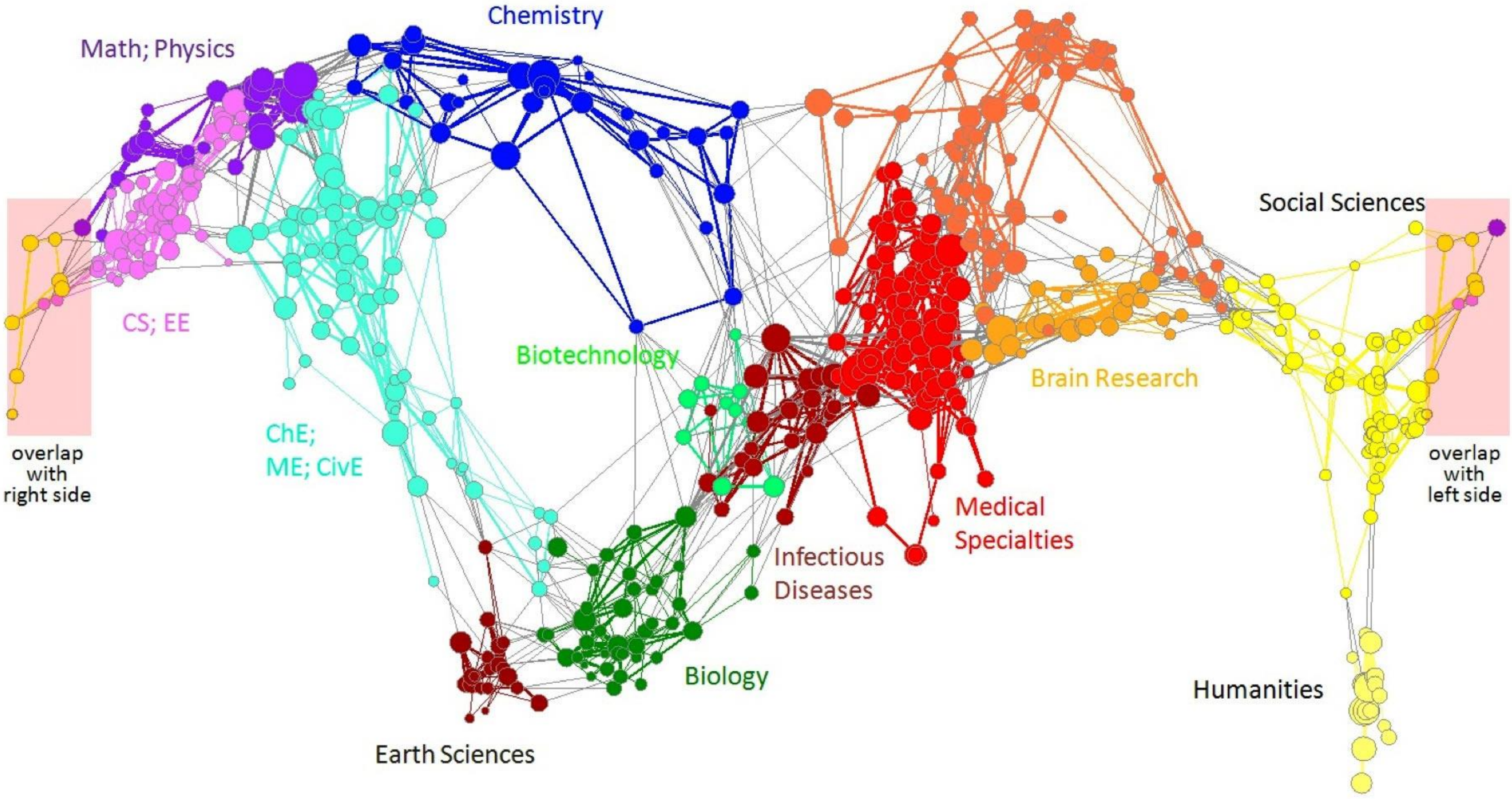
Graph Data: Media Networks



Connections between political blogs

Polarization of the network [Adamic-Glance, 2005]

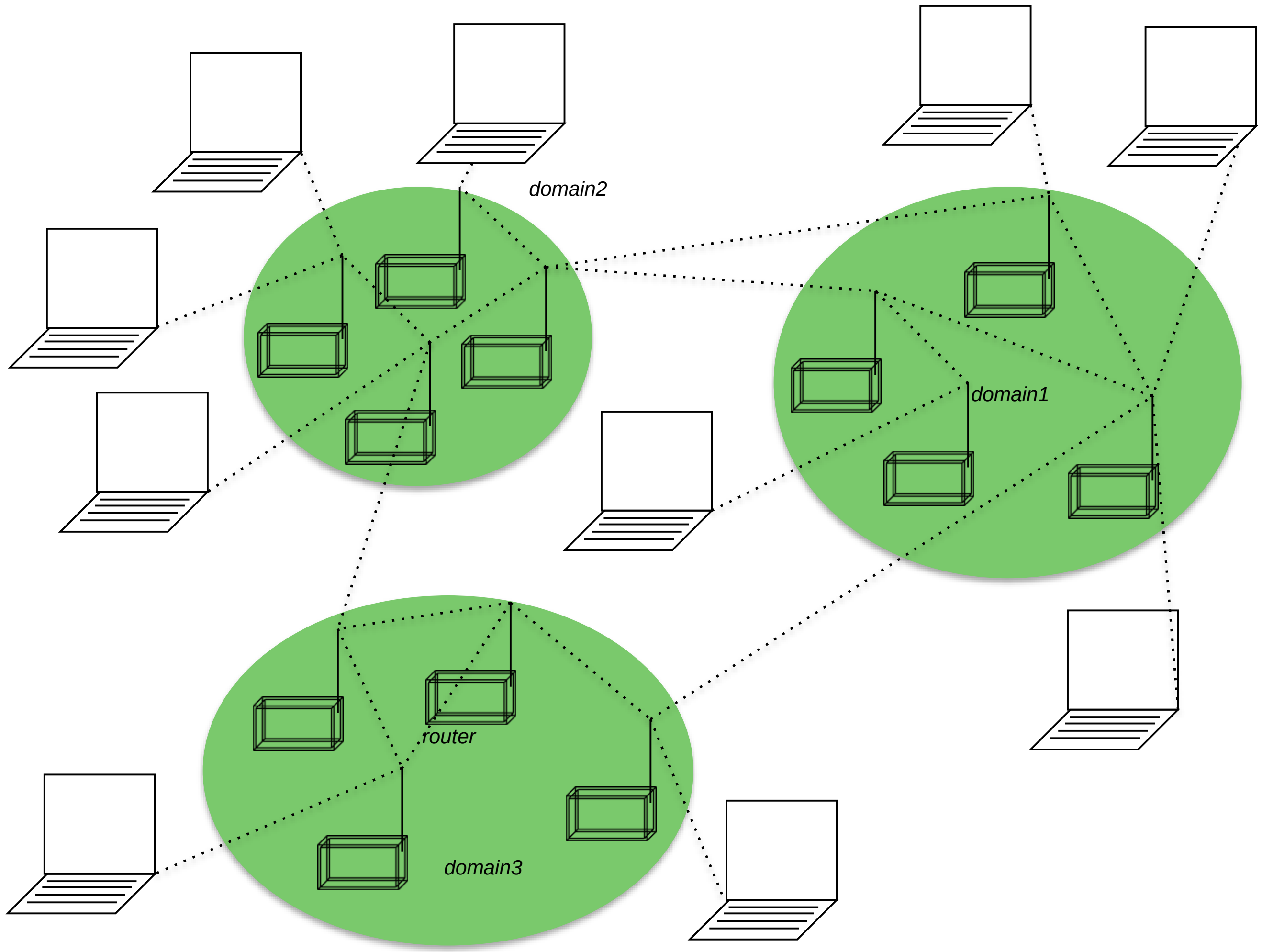
Graph Data: Information Nets



Citation networks and Maps of science

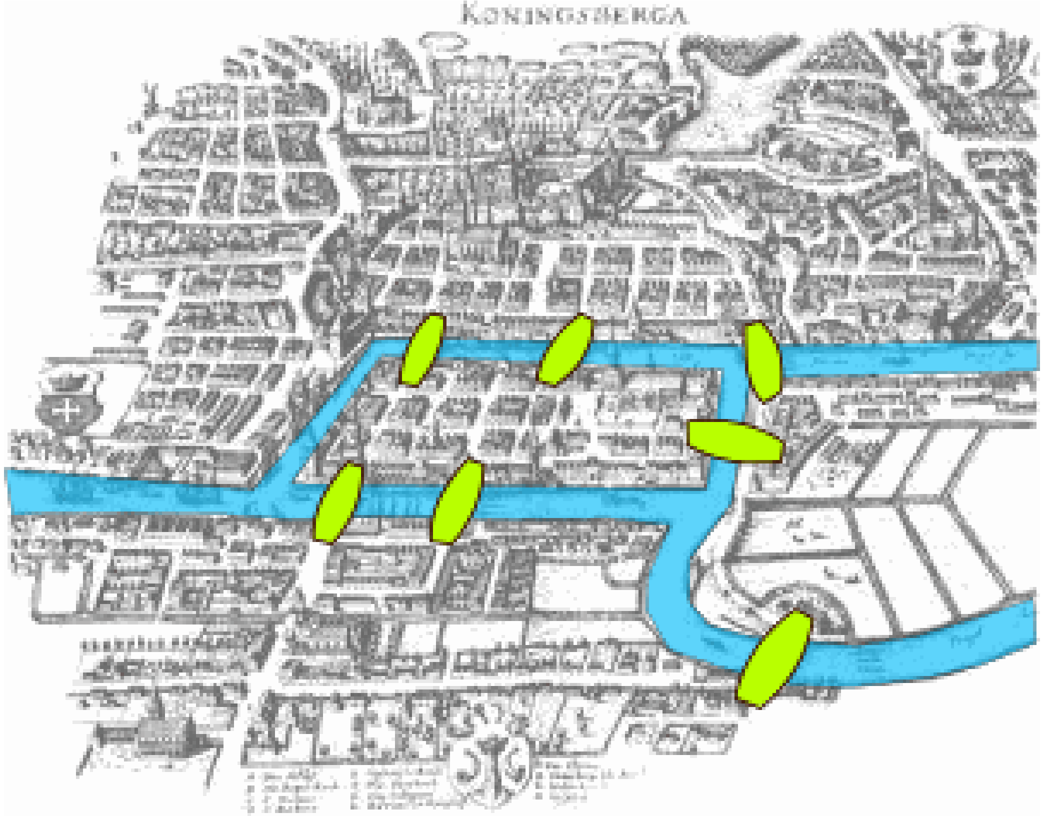
[Börner et al., 2012]

Graph Data: Communication Nets



Internet

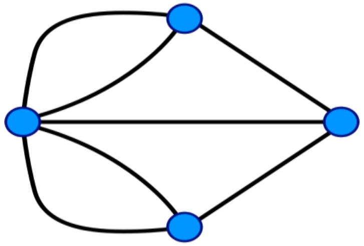
Graph Data: Classic Example



Seven Bridges of Königsberg

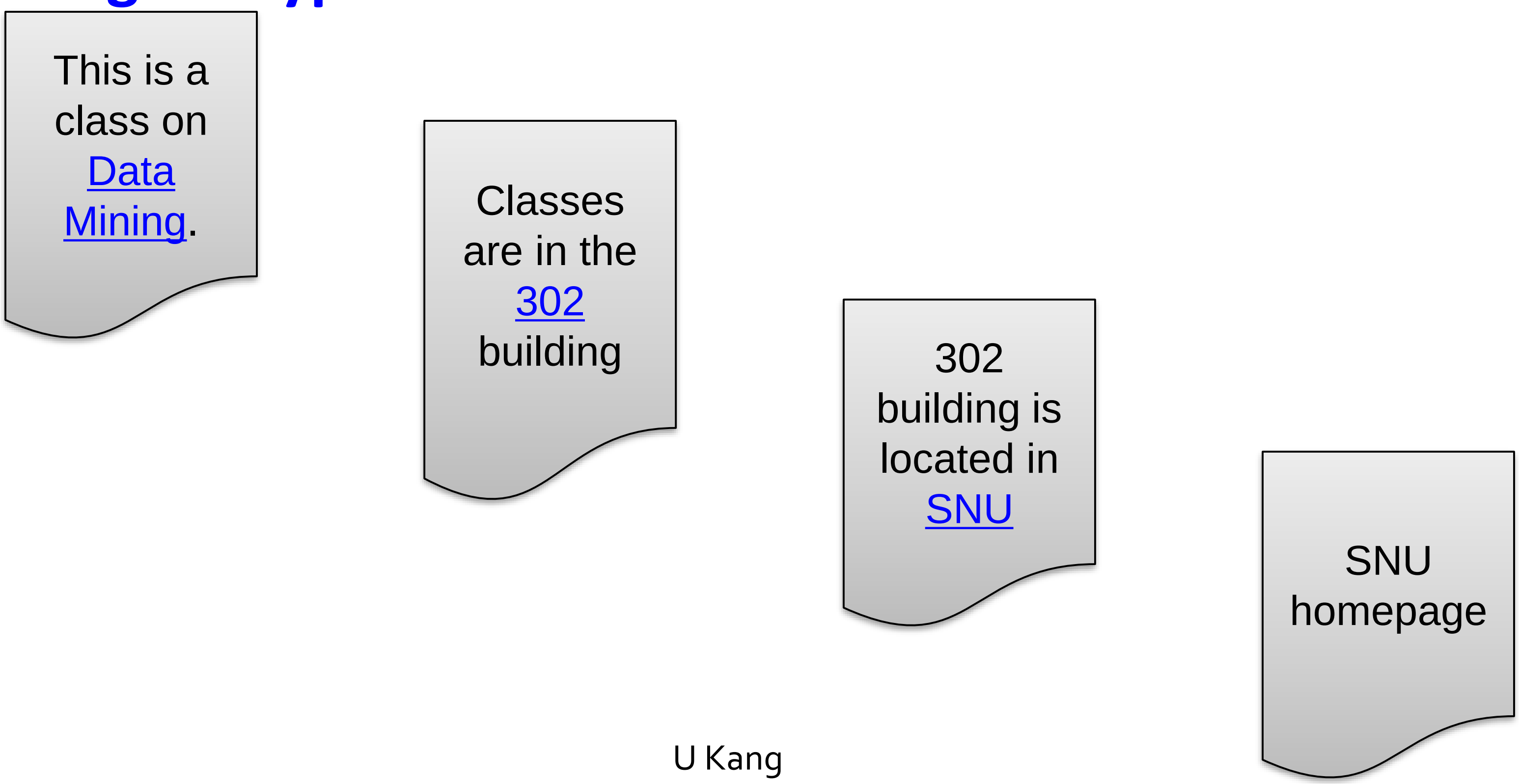
[Euler, 1735]

Return to the starting point by traveling each link of the graph once and only once.



Web as a Graph

- **Web as a directed graph:**
  - **Nodes: Webpages**
  - **Edges: Hyperlinks**



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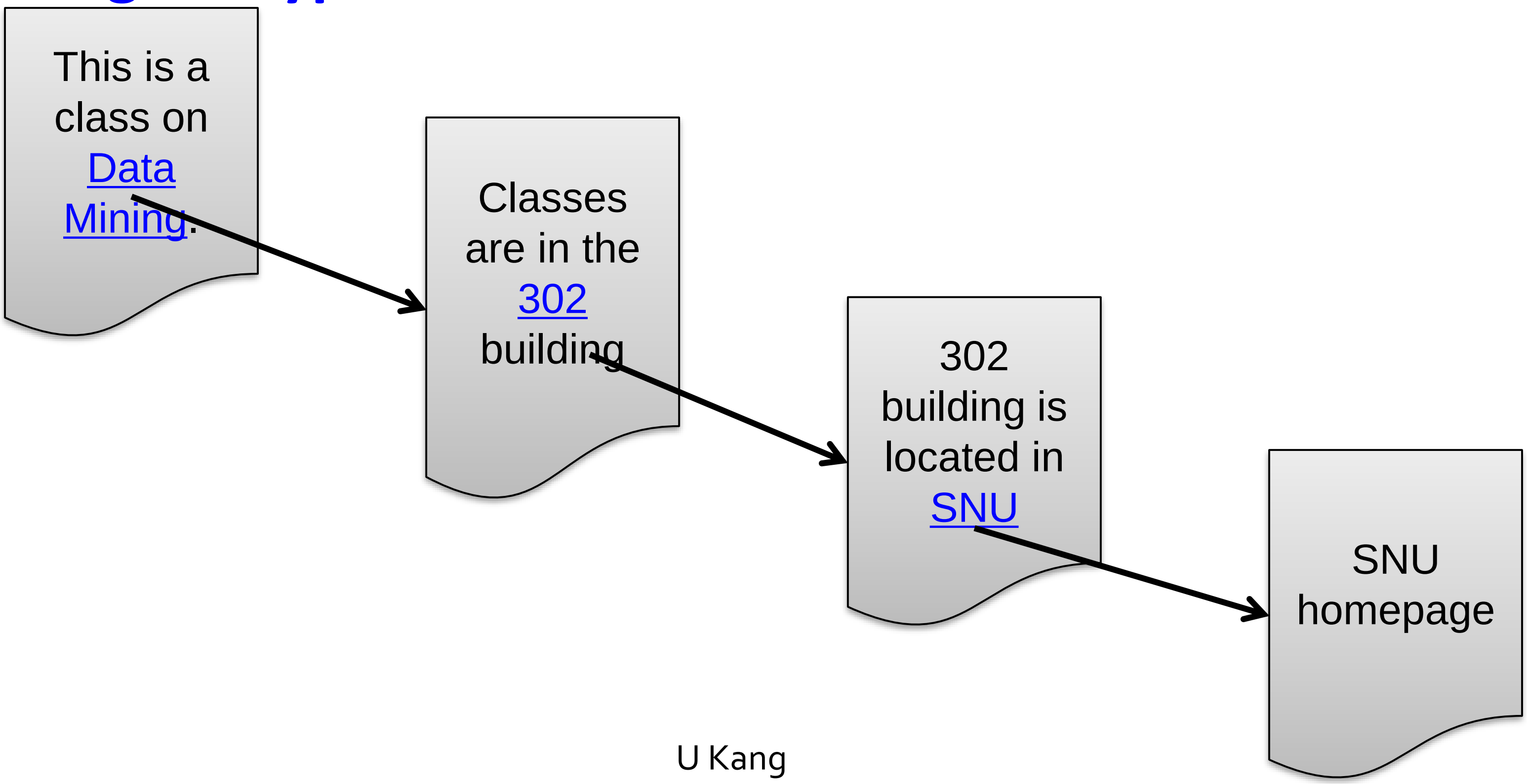
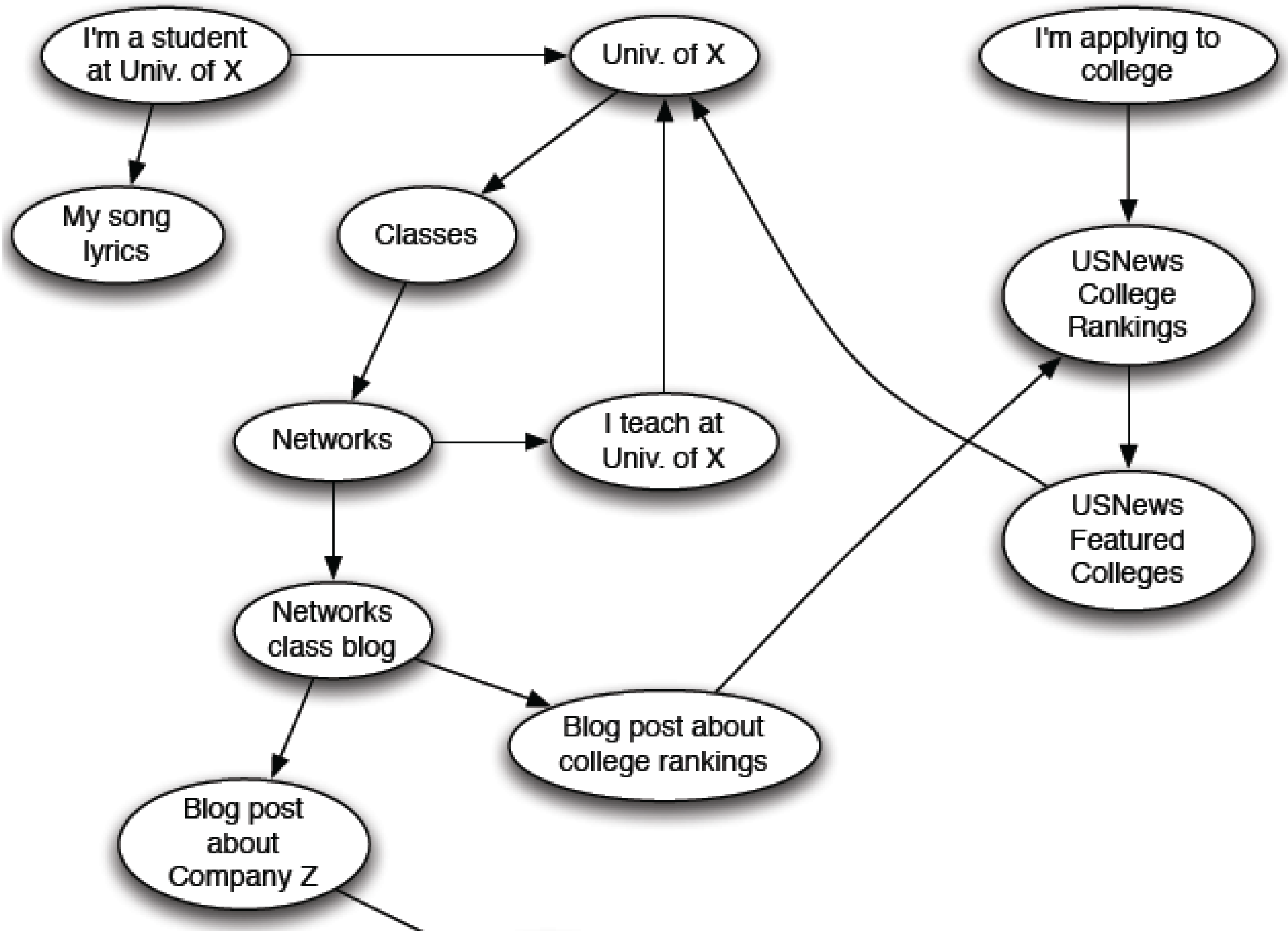


Web as a Directed Graph



Broad Question

- **How to organize the Web?**
- **First try:** Human curated **Web directories**
  - Yahoo, DMOZ, LookSmart
- **Second try: Web Search**
  - **Information Retrieval** investigates: find relevant docs in a small and trusted set
    - Newspaper articles, Patents, etc.
  - **But:** Web is **huge**, full of untrusted documents, random things, web spam, etc.

Web Search: 2 Challenges

2 challenges of web search:

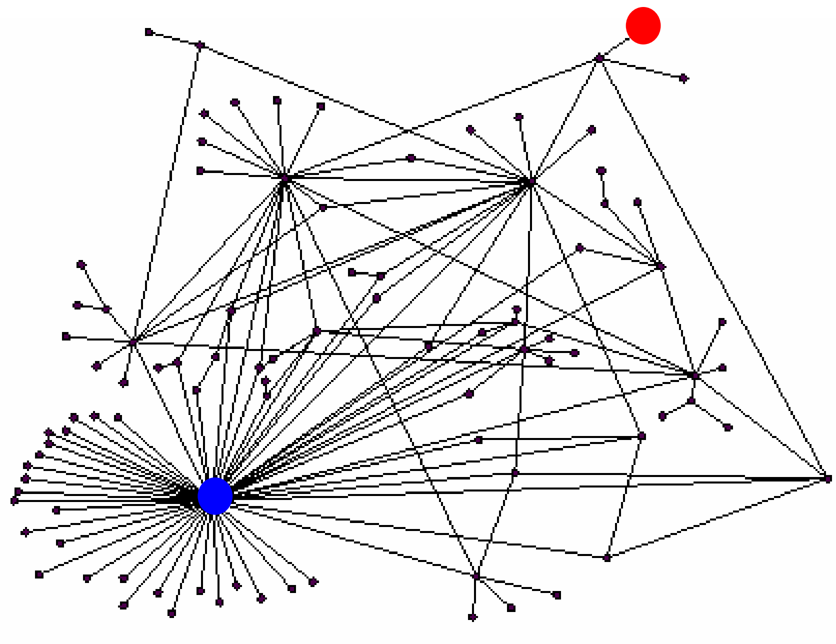
- **(1) Web contains many sources of information Who to "trust"?**
  - **Idea:** Trustworthy pages may point to each other!
- **(2) What is the "best" answer to the query "newspaper"?**
  - No single right answer
  - **Idea:** Pages that actually know about newspapers might all be pointing to many newspapers

Ranking Nodes on the Graph

- **All web pages are not equally "important"**

  www.joe-schmoe.com vs. www.snu.ac.kr

- There is large diversity in the web-graph node connectivity. **Let's rank the pages by the link structure!**



Link Analysis Algorithms

- We will cover the following **Link Analysis approaches** for computing **importance** of nodes in a graph:
  - Page Rank
  - Topic-Specific (Personalized) Page Rank
  - Web Spam Detection Algorithms

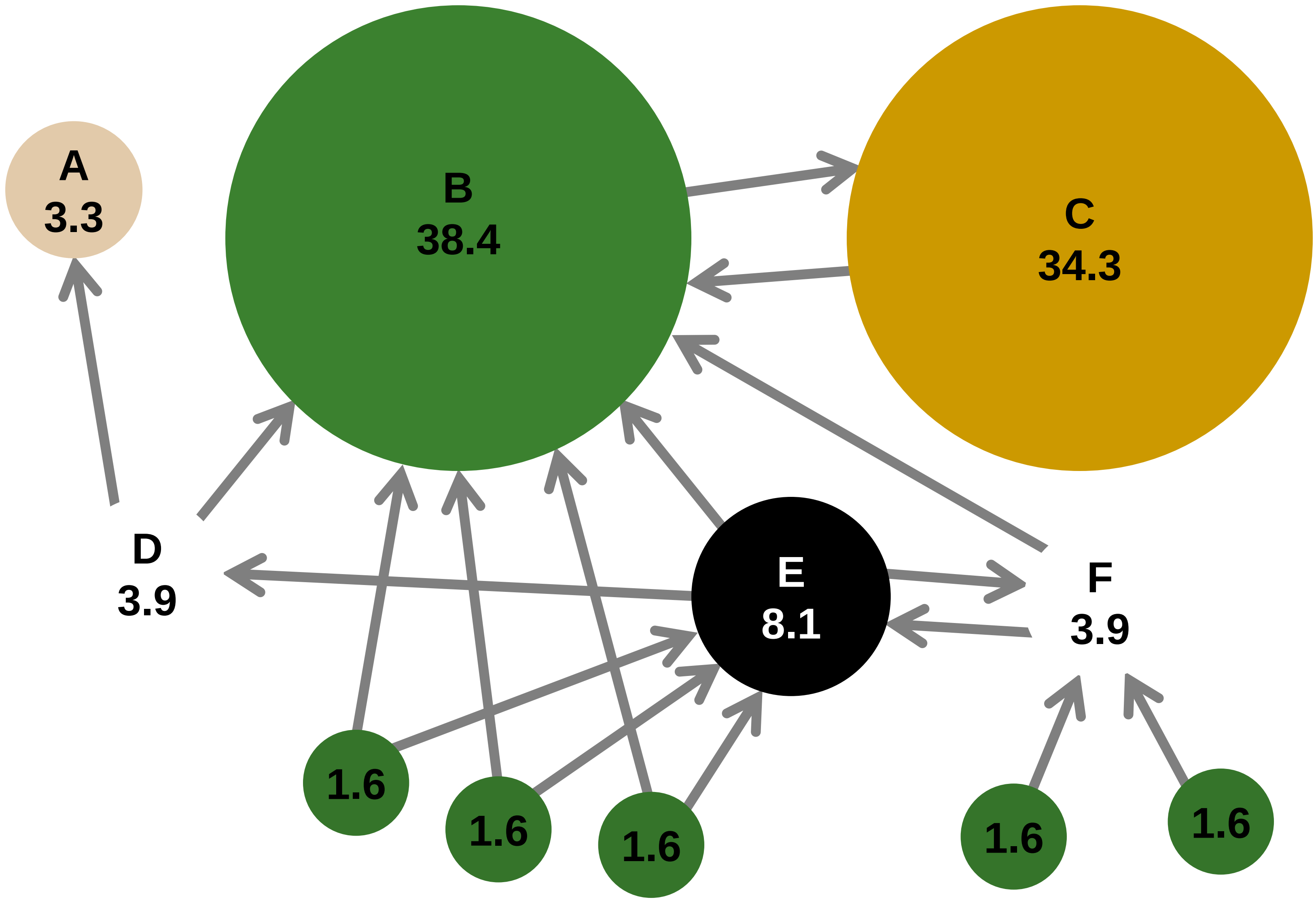
Outline

- ☑ Overview
- ☐ **PageRank: Flow Formulation**

Links as Votes

- **Idea: Links as votes**
  - **A page is more important if it has more links**
    - In-coming links? Out-going links?
- **Think of in-links as votes:**
  - www.snu.ac.kr has 100,000 in-links
  - www.joe-schmoe.com has 1 in-link
- **Are all in-links equal?**
  - **Links from important pages count more**
  - Recursive question!

Example: PageRank Scores



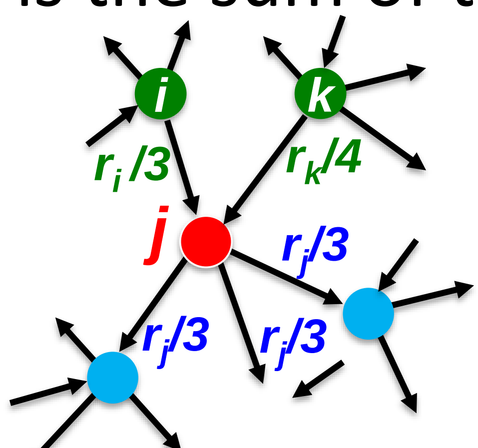
Simple Recursive Formulation

- Each link's vote is proportional to the **importance** of its source page

- If page j with importance r_j has n out-links, each link gets r_j / n votes

- Page j's own importance is the sum of the votes on its in-links

$$r_j = r_i/3 + r_k/4$$

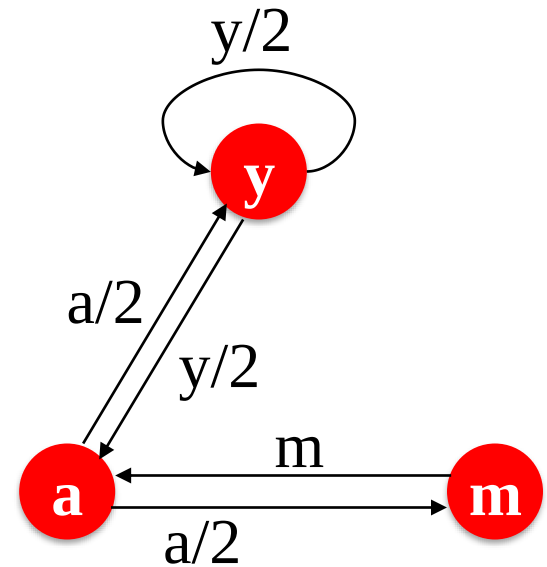


PageRank: The "Flow" Model

- A "vote" from an important page is worth more
- A page is important if it is pointed to by other important pages
- Define a "rank" r_j for page j

$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_i}$$

d_i … out-degree of node i
i -> j : all i that point to j



"Flow" equations:

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

$r_m = r_a/2$

Solving the Flow Equations

Flow equations:

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

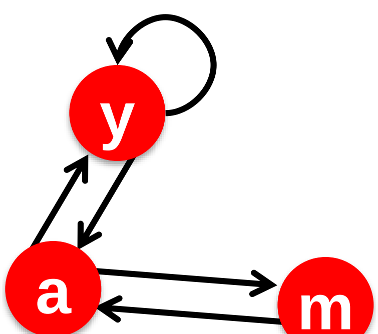
$r_m = r_a/2$

- **3 equations, 3 unknowns, no constants**
  - No unique solution
  - All solutions equivalent modulo the scale factor
    - I.e., Multiplying c to given a solution r_y, r_a, r_m will give you another solution
- **Additional constraint forces uniqueness:**
  - $\boldsymbol{r_y + r_a + r_m = 1}$
  - **Solution:** $\boldsymbol{r_y = \frac{2}{5},\ r_a = \frac{2}{5},\ r_m = \frac{1}{5}}$
- **Gaussian elimination method works for small examples, but we need a better method for large web-size graphs**
- **We need a new formulation!**

PageRank: Matrix Formulation

- **Stochastic adjacency matrix *M***
  - Let page i has d_i out-links
  - If $i \rightarrow j$, then $M_{ji} = \frac{1}{d_i}$ else $M_{ji} = 0$
    - ***M*** is a **column stochastic matrix**
      - Each column sums to 1

NOTE: *A matrix **M** is called `column stochastic' if the sum of each column is 1*



Source (columns) / Destination (rows)

	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

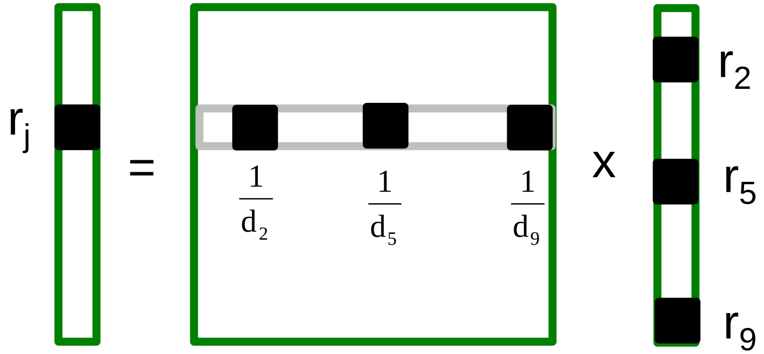
M

PageRank: Matrix Formulation

- **Rank vector r:** vector with an entry per page
  - r_i is the importance score of page i
  - $\sum_i r_i = 1$
- **The flow equations** $r_j = \sum_{i \to j} \frac{r_i}{d_i}$ **can be written**

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

Why?



Eigenvector Formulation

- **The flow equations can be written**

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

- So the **rank vector** ***r*** is an **eigenvector** of the web matrix ***M*,** with the corresponding eigenvalue 1

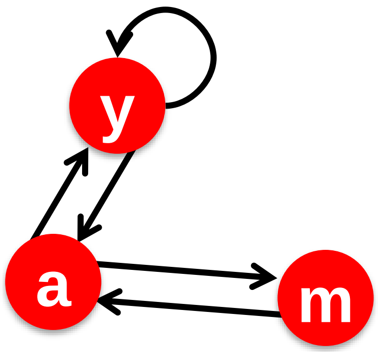
- **Fact**: The largest eigenvalue of a column stochastic matrix is 1

- **We can now efficiently solve for *r*!**
  The method is called Power iteration

NOTE: ***x*** is an eigenvector with the corresponding eigenvalue **λ** if:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$$

Example: Flow Equations & M



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$$r = M \cdot r$$

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} y \\ a \\ m \end{bmatrix} = \begin{bmatrix} ½ & ½ & 0 \\ ½ & 0 & 1 \\ 0 & ½ & 0 \end{bmatrix} \begin{bmatrix} y \\ a \\ m \end{bmatrix}$$

Power Iteration Method

- **Given a web graph with *n* nodes, where the nodes are pages and edges are hyperlinks**
- **Power iteration:** a simple iterative scheme
  - Suppose there are *N* web pages
  - Initialize: $\mathbf{r}^{(0)} = [1/N,\ldots.,1/N]^T$
  - Iterate: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$
  - Stop when $|\mathbf{r}^{(t+1)} - \mathbf{r}^{(t)}|_1 < \varepsilon$

    $|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ (called the $\mathbf{L_1}$ norm)

    Can use any other vector norm, e.g., Euclidean

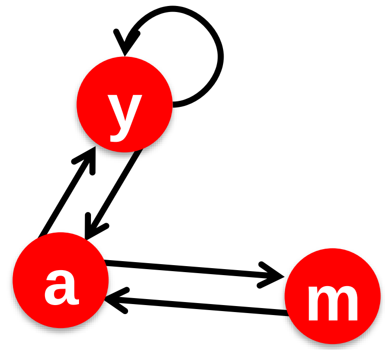
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d}_i}$$

d_i …. out-degree of node i

Power Iteration Method

- **Power iteration:**
  A method for finding dominant eigenvector (the vector corresponding to the largest eigenvalue)
  - $\boldsymbol{r}^{(1)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(0)}$
  - $\boldsymbol{r}^{(2)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(1)} = \boldsymbol{M}\left(\boldsymbol{M}\boldsymbol{r}^{(1)}\right) = \boldsymbol{M}^2 \cdot \boldsymbol{r}^{(0)}$
  - $\boldsymbol{r}^{(3)} = \boldsymbol{M} \cdot \boldsymbol{r}^{(2)} = \boldsymbol{M}\left(\boldsymbol{M}^2\boldsymbol{r}^{(0)}\right) = \boldsymbol{M}^3 \cdot \boldsymbol{r}^{(0)}$
- **Fact:**
  Sequence $\boldsymbol{M} \cdot \boldsymbol{r}^{(0)}, \boldsymbol{M}^2 \cdot \boldsymbol{r}^{(0)}, \ldots \boldsymbol{M}^k \cdot \boldsymbol{r}^{(0)}, \ldots$ approaches the dominant eigenvector of $\boldsymbol{M}$
  - Dominant eigenvector = the one corresponding to the largest eigenvalue

PageRank: How to solve?



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$\mathbf{r_y} = \mathbf{r_y}/2 + \mathbf{r_a}/2$

$\mathbf{r_a} = \mathbf{r_y}/2 + \mathbf{r_m}$

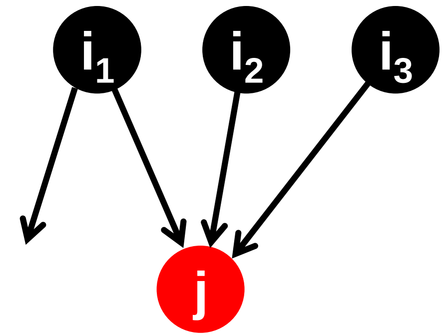
$\mathbf{r_m} = \mathbf{r_a}/2$

- **Power Iteration:**
  - Set $r_j = 1/N$
  - **1:** $r'_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - **2:** $r = r'$
  - Goto **1**
- **Example:**

$$\begin{pmatrix} r_y \\ r_a \\ r_m \end{pmatrix} = \begin{matrix} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{matrix}$$

Iteration 0, 1, 2, …

Random Walk Interpretation



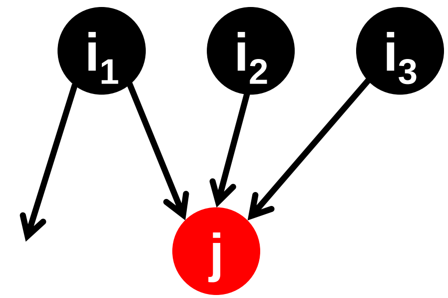
$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_{\mathrm{out}}(\mathrm{i})}$$

- **Imagine a random web surfer:**
  - At any time $\boldsymbol{t}$, surfer is on some page $\boldsymbol{i}$
  - At time $\boldsymbol{t+1}$, the surfer follows an out-link from $\boldsymbol{i}$ uniformly at random
  - Ends up on some page $\boldsymbol{j}$ linked from $\boldsymbol{i}$
  - Process repeats indefinitely
- **Let:**
  - $\boldsymbol{p}(\boldsymbol{t})$ … vector whose $\boldsymbol{i}^{\text{th}}$ coordinate is the prob. that the surfer is at page $\boldsymbol{i}$ at time $\boldsymbol{t}$
  - So, $\boldsymbol{p}(\boldsymbol{t})$ is a probability distribution over pages

The Stationary Distribution

- **Where is the surfer at time *t+1*?**
  - Follows a link uniformly at random

$$\boldsymbol{p}(\boldsymbol{t}+\mathbf{1}) = \boldsymbol{M} \cdot \boldsymbol{p}(\boldsymbol{t})$$



$$p(t+1) = \mathrm{M} \cdot p(t)$$

- Suppose the random walk reaches a state

$$\boldsymbol{p}(\boldsymbol{t}+\mathbf{1}) = \boldsymbol{M} \cdot \boldsymbol{p}(\boldsymbol{t}) = \boldsymbol{p}(\boldsymbol{t})$$

then $\boldsymbol{p}(\boldsymbol{t})$ is called **stationary distribution** of a random walk

- **Our original rank vector** $\boldsymbol{r}$ satisfies $\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$
  - **So, $\boldsymbol{r}$ is a stationary distribution for the random walk**

Existence and Uniqueness

- **A central result from the theory of random walks (a.k.a. Markov processes):**

For graphs that satisfy **certain conditions**, the **stationary distribution is unique** and eventually will be reached no matter what the initial probability distribution at time **t = 0**

Certain conditions: a walk starting from a random page can reach any other page, and the graph is not bipartite

What You Need to Know

- Motivation for link analysis
  - Graphs are everywhere
  - Web as graphs
- Pagerank: an important graph ranking algorithm
  - A page is important if it is pointed to by other important pages
  - Pagerank vector gives the stationary distribution for the random walk on a graph

Questions?